POSITIVE PRESENTATIONS OF THE BRAID GROUPS AND THE EMBEDDING PROBLEM

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Abstract. A large class of positive finite presentations of the braid groups is found and studied. It is shown that no presentations but known exceptions in this class have the property that equivalent braid words are also equivalent under positive relations.

1. Introduction

We will discover a large class of positive finite presentations of the n-braid group. Roughly speaking, a presentation in this class has a set of generators determined by a connected graph with n vertices immersed in a plane in such a way that each pair of edges intersects at most once. Our class of presentations includes all known positive finite presentations such as the Artin presentation[1], the band-generator presentation in [3, 9] and all of Sergiescu's presentations in [11]. Our first objective is to find a (minimal) collection of positive relations among generators given by a graph as above so that it becomes a presentation of the n-braid groups.

The semigroup of all positive words plays a crucial role in the solutions to the word problem and the conjugacy problem for the n-braid group given by Garside[7], Thurston[6], and Elrifi-Morton[5]. Their solutions are based on the property that the semigroup embeds in the whole group. Birman-Ko-Lee's recent work[3] also requires this embedding property in order to give a fast solution to the word problem in the band-generator presentation. Our second objective is to prove that with a few exception all presentations in our class have the embedding property. The Artin and the band-generator presentations are the only nontrivial exceptions. Consequently these two among all presentations are natural and interesting for further study.

2. Immersed graphs and known presentations

The n-braid group B_n is the group of isotopy classes of orientation-preserving automorphisms of an n punctured plane $\mathbf{R}^2$ that fix the set of punctures and the outside of a disk containing all punctures. It is customary to place punctures on the x-axis and equally spaced. Ignoring the punctures, such an automorphism is a homeomorphism of $\mathbf{R}^2$ permuting the punctures and so is isotopic to the identity map of $\mathbf{R}^2$. A

geometric braid is determined by taking the trace of the punctures under this isotopy. When the punctures are not in the customary location, there is a choice of homeomorphisms of $\mathbf{R}^2$ that send the punctures to their customary location and a geometric braid is uniquely determined up to conjugation in this case.

Since the symmetric group over n letters is naturally a quotient of B_n, an automorphism that exchanges two punctures is a candidate for a generator of B_n and a typical automorphism of this type can be easily depicted as an arc connecting the two punctures so the automorphism exchanges two punctures clockwise or counterclockwise inside an annulus lying along the arc. See Figure 1 for an example. This type of braids will be called the *half twist along the given arc*.

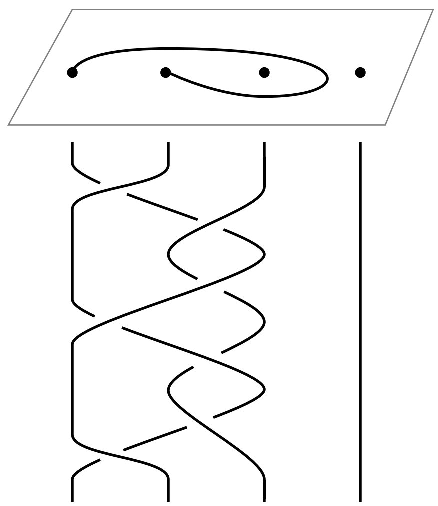

FIGURE 1.

Therefore by regarding the punctures as vertices and a collection of these arcs as edges, we have a graph Γ immersed in $\mathbf{R}^2$ satisfying the following properties:

1. There are no loops;
2. Edges have no self-intersections;
3. Two distinct edges intersect either at interior points transversely or at common end points.
4. There is at least a vertex inside a "pseudo digon", that is, a region cobounded by two subarcs from two edges so that corners of the region are either two interior intersection points or one vertex and one interior intersection point.

If the number of intersections among edges is minimized, for example, by making all edges geodesics in the hyperbolic structure of the punctured plane, the condition (4) holds automatically. Two immersed graphs are regarded as equivalent if one graph can be transformed to the other by an orientation-preserving homeomorphism of $\mathbf{R}^2$. Thus we may assume that vertices lie on the customary locations if necessary.

Throughout this article, graphs mean immersed graphs in $\mathbf{R}^2$ satisfying the condition discussed. If graphs are embedded in $\mathbf{R}^2$, we call them *planar*. The set of edges in a graph Γ with n vertices corresponds to a set of elements in B_n as in Figure 1. Two equivalent graphs determine two sets of braids that differ by an inner automorphism of B_n. If the set corresponding to edges generates B_n, we say that Γ *generates* B_n and we are interested in graphs that generate B_n. Since an edge of Γ corresponds to a transposition in the symmetric group, a graph that generates B_n must be connected in order to permute any two vertices. But there are connected graphs that do not generate B_n. The problem of deciding when an immersed graph generates the braid group is considered elsewhere[8].

We now discuss how a braid word corresponds to an edge of a graph with n vertices in the customary location. The Artin generator σ_i(or σ_i^{-1}) for $i = 1, \dots, n-1$ corresponds to the clockwise(counterclockwise, respectively) half twist along the straight edge joining the i-th and the $(i+1)$-st vertices. Given an edge of a graph, we simplify the edge via a sequence of half twists σ_i or σ_i^{-1} for $i = 1, \dots, n-1$ until it becomes a straight edge corresponding to, say, σ_k. The required sequence can be expressed as a word W in Artin generators in which a half twist applied later is written on the left. Then the edge that we started with corresponds to the braid word $W^{-1}\sigma_k W$. The word W is not uniquely expressed but W as an element of B_n is well-defined. The inverse of this correspondence is similar. Given a conjugate $W^{-1}\sigma_k W$, apply a sequence of half twists determined from right to left by W^{-1} to the straight edge σ_k. Consequently each edge of a graph with n vertices is uniquely represented by an element in B_n that can be written as a conjugate of an Artin generator. The following lemma summarizes this discussion.

Lemma 2.1. *A braid β in B_n is the half twist along an arc joining two punctures in the plane if and only if β can be written as a conjugate of an Artin generator or its inverse.*

From now on as long as no confusion arises, we will not distinguish three concepts, namely an edge of a graph, the half twist along the edge, and a conjugate word expressing the edge. Since any two Artin generators are conjugate each other, any half twist along an arc corresponds to a conjugate of a fixed Artin generator. The following is immediate from the above lemma.

Corollary 2.2. *A graph Γ of n vertices generates B_n if and only if each Artin generator can be expressed as $W\alpha W^{-1}$ for some edge α and some word W on edges of Γ.*

In the view of the above corollary, it is important to know how the conjugate of an edge by another edge in a graph looks like. Let α and β be words in B_n that express edges of a graph. If the edges α and β

do not intersect each other and are not adjacent, they commute and so $\alpha\beta\alpha^{-1} = \beta = \alpha^{-1}\beta\alpha$. If the edges α and β do not intersect each other and are adjacent, the three edges α, β, and $\alpha\beta\alpha^{-1} = \beta^{-1}\alpha\beta$ form a triangle counterclockwise and the three edges α, β, and $\alpha^{-1}\beta\alpha = \beta\alpha\beta^{-1}$ form a triangle clockwise as in Figure 2. We note that there should be no other vertex inside the triangles and this can be achieved by drawing thin triangles inside a sufficiently small neighborhood of the union of two edges α and β. When the edges α and β intersect at an interior point, one can still describe the edges $\alpha\beta\alpha^{-1}$ and $\alpha^{-1}\beta\alpha$ but we do not need them in this article.

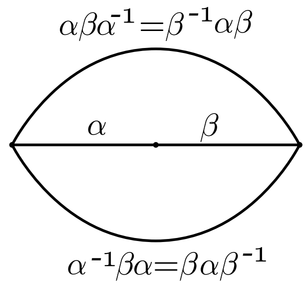

FIGURE 2.

Give a set E of generators of the n-braid group B_n, a *positive word* in E is a product of positive powers of generators in E. A presentation of a group is *finite* if there are finitely many generators and relations. A presentation is *positive* if all defining relations are equations of positive words in generators.

We now introduce the graphs that corresponds to some of known positive finite presentations of the braid groups. The set of Artin generators $\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_{n-1}$ of B_n forms the graph in Figure 3 and a minimal set of defining relations is given by

$$\begin{aligned} \sigma_i\sigma_j\sigma_i &= \sigma_j\sigma_i\sigma_j && \text{if} \quad |i-j| = 1 \\ \sigma_i\sigma_j &= \sigma_j\sigma_i && \text{if} \quad |i-j| > 1. \end{aligned}$$

FIGURE 3.

The graphs in Figure 3 will be called the *Artin* graph.

Sergiescu [11] showed how a finite presentation of B_n can be obtained from any planar connected graph with n vertices. He described a sufficient set of positive relations that depend only on the geometry of a given planar graph. We will give a minimal set of positive relations as a corollary of the main theorem in §3. Let n be the number of vertices in Γ. By choosing a fixed n points in $\mathbf{R}^2$ and a homeomorphism of $\mathbf{R}^2$ that sends vertices of Γ to the n fixed points, the group B_Γ introduced in [11] is identified with the n-braid group B_n. These presentations of

B_n will be called *Sergiescu's presentations.* The relations in a Sergiescu's presentation are highly redundant but they are useful when we need to find a relation locally given by a planar graph.

Recently a new presentation of B_n called the band-generator presentation has been developed by Birman-Ko-Lee[3]. This presentation has $\binom{n}{2}$ generators a_{ts} for $1 \leq s < t \leq n$ corresponds to the graph in Figure 4 and defining relations:

$$\begin{array}{cl} a_{ts}a_{sr} = a_{tr}a_{ts} = a_{sr}a_{tr} & \text{for all } t, s, r \text{ with } \ n \geq t > s > r \geq 1 \\ a_{ts}a_{rq} = a_{rq}a_{ts} & \text{if } \ (t-r)(t-q)(s-r)(s-q) > 0. \end{array}$$

The Artin generator and the band-generators are related as

$$a_{ts} = (\sigma_{t-1}\sigma_{t-2}\cdots\sigma_{s+1})\sigma_s(\sigma_{s+1}^{-1}\cdots\sigma_{t-2}^{-1}\sigma_{t-1}^{-1})$$

and

$$\sigma_t = a_{(t+1)t}.$$

The set of band-generators are depicted as the graph in Figure 4 that will be called the *inner complete* graph.

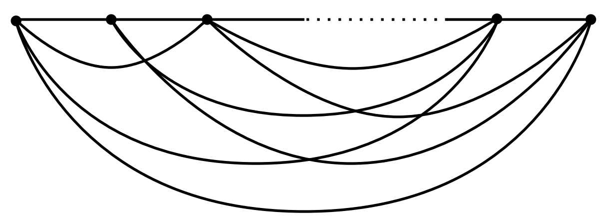

FIGURE 4.

The relations $a_{ts}a_{sr} = a_{tr}a_{ts} = a_{sr}a_{tr}$ will be called a *triangular* relation. A triangular relation is derived whenever a new generator is introduced by means of a conjugation of an edge by an adjacent edge. Triangular relations serve as building blocks of positive relations in the braid groups.

3. POSITIVE PRESENTATIONS FROM LINEARLY SPANNED GRAPHS

After an immersed graph is turned into a planar graph by regarding all interior intersections as vertices, a region bounded by a closed edge-path is called a *pseudo face* if the edge-path contains at least one vertex that is an interior intersection of two edges as in Figure 5.

A graph Γ is said to be *linearly spanned* if it is connected and there is no vertex in any pseudo face of Γ. A connected subgraph of a linearly spanned graph is clearly linearly spanned. Two edges of a linearly spanned graph intersect each other at most once since any pseudo digon cobounded by two edges can not exist in a linearly spanned graph. All planar graphs and all subgraphs of the inner-complete graph are linearly spanned. For example the graph on the left in Figure 6 is

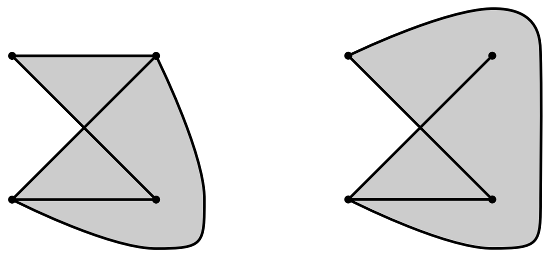

FIGURE 5.

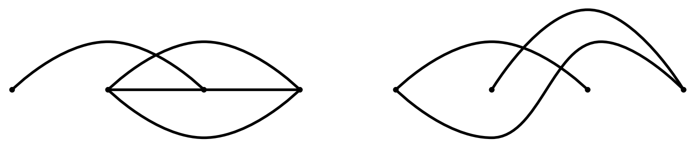

FIGURE 6.

neither planar nor a subgraph of the inner-complete graph but it is linearly spanned, and the graph on the right is not linearly spanned

The following lemma justifies the terminology "linearly spanned".

Lemma 3.1. *A linearly spanned graph that is a tree is equivalent to a subgraph of the inner-complete graph.*

Proof. Choose a point x far away from the graph. Since the graph is a tree and no vertices are surrounded by edges, there are n arcs that join x to each vertex and are disjoint each other and are disjoint from the graph as in Figure 7. Choose a new horizontal axis disjoint from the graph and move each vertex along each arc by a homeomorphism so that it lies on the new axis as Figure 7. Then the result is a subgraph of the inner complete graph.

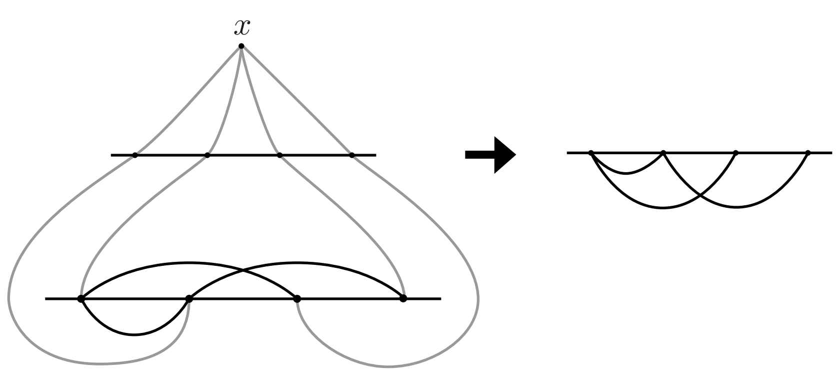

FIGURE 7.

□

Corollary 3.2. *A linearly spanned graph* Γ *with* n *vertices generates* B_n.

Proof. A maximal tree of Γ is equivalent to a connected subgraph of the inner-complete graph from which we can obtain the inner complete graph by adding missing edges that are conjugates of an existing edge by an adjacent edge. $\square$

Lemma 3.3. *Let T be a linearly spanned tree with n vertices. Then T generates B_n with $\frac{(n-1)(n-2)}{2}$ positive relations described as in the proof below.*

Proof. Induction on n. It is trivial when $n = 2$. Suppose a linear spanned tree T' with $n-1$ vertices generates B_{n-1} with $\frac{(n-2)(n-3)}{2}$ positive relations. We add a new vertex and a new edge α to T'. For each edge β of T', we will have a new positive relation so that $n-2$ new positive relations will be added. In the following we use Tietze transformations [12] that add and delete a generator(s) denoted by λ or μ to utilize triangular relations.

1. If α and β have no intersection, add the positive relation
$$\alpha\beta = \beta\alpha.$$
2. For each set of edges $\beta_1, \ldots, \beta_m$ simultaneously adjacent to α at a vertex of valency $m+1$ as in Figure 8(a), add m positive relations
$$\begin{aligned} \alpha\beta_m\alpha &= \beta_m\alpha\beta_m \\ \alpha\beta_1\beta_m\alpha &= \beta_m\alpha\beta_1\beta_m \\ &\vdots \\ \alpha\beta_{m-1}\beta_m\alpha &= \beta_m\alpha\beta_{m-1}\beta_m \end{aligned}$$
3. If α and β intersect and form a pseudo face as in Figure 8(b), add the positive relation
$$\begin{aligned} &\beta_1\beta_2\cdots\beta_m\alpha\beta\beta_1^2\beta_2\cdots\beta_m\alpha\beta\beta_1\beta_2\cdots\beta_{m-1} \\ &= \beta_2\beta_3\cdots\beta_m\alpha\beta\beta_1^2\beta_2\cdots\beta_m\alpha\beta\beta_1\beta_2\cdots\beta_m \end{aligned}$$
that is derived from $\beta\lambda\beta = \lambda\beta\lambda$ where
$$\begin{aligned} \lambda &= \beta_1\beta_2\cdots\beta_m\alpha\beta_m^{-1}\cdots\beta_2^{-1}\beta_1^{-1} \\ &= \beta^{-1}\beta_m^{-1}\cdots\beta_2^{-1}\beta_1\beta_2\cdots\beta_m\alpha \end{aligned}$$
4. If α and β intersect and form a pseudo face as in Figure 9(a), add the positive relation
$$\beta_1\beta_2\cdots\beta_m\alpha\beta\beta_1\beta_2\cdots\beta_m = \beta_2\beta_3\cdots\beta_m\alpha\beta\beta_1\beta_2\cdots\beta_m\alpha$$
that is derived from $\beta\lambda = \lambda\beta$ where
$$\begin{aligned} \lambda &= \beta_1\beta_2\cdots\beta_m\alpha\beta_m^{-1}\cdots\beta_2^{-1}\beta_1^{-1} \\ &= \beta^{-1}\beta_m^{-1}\cdots\beta_2^{-1}\beta_1\beta_2\cdots\beta_m\alpha \end{aligned}$$

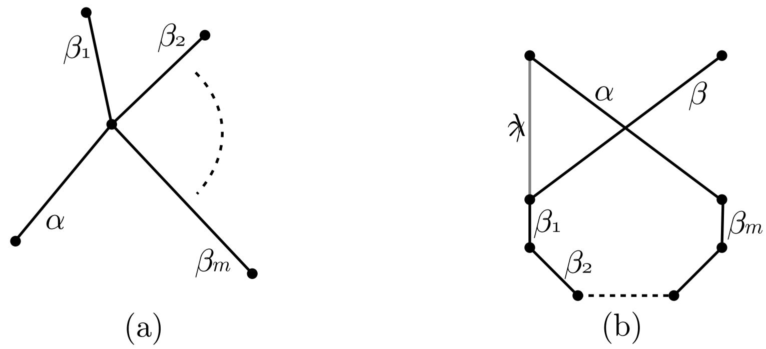

FIGURE 8.

5. If α and β intersect and form a pseudo face as in Figure 9(b), add the positive relation

$$\begin{aligned}&\gamma_2\cdots\gamma_\ell\alpha\beta\beta_1\cdots\beta_m\gamma_1\cdots\gamma_\ell\alpha\beta\beta_1\cdots\beta_{m-1}\\ =\ &\beta_1\cdots\beta_m\gamma_1\cdots\gamma_\ell\alpha\beta\beta_1\cdots\beta_m\gamma_1\cdots\gamma_\ell\end{aligned}$$

that is derived from $\lambda\mu=\mu\lambda$ where

$$\begin{aligned}\lambda &= \beta\beta_1\beta_2\cdots\beta_m\beta_{m-1}^{-1}\cdots\beta_2^{-1}\beta_1^{-1}\beta^{-1}\\ &= \beta_m^{-1}\cdots\beta_1^{-1}\beta\beta_1\beta_2\cdots\beta_m\\ \mu &= \gamma_1\gamma_2\cdots\gamma_\ell\alpha\gamma_\ell^{-1}\cdots\gamma_2^{-1}\gamma_1^{-1}\\ &= \alpha^{-1}\gamma_\ell^{-1}\cdots\gamma_2^{-1}\gamma_1\gamma_2\cdots\gamma_\ell\alpha\end{aligned}$$

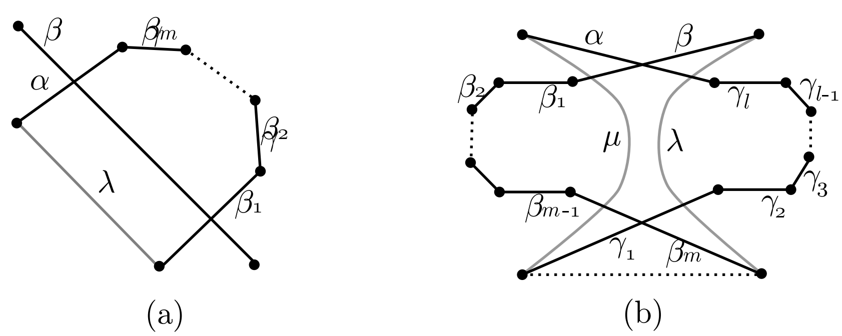

FIGURE 9.

□

The following is the main theorem of this section and the known presentation mentioned in the previous section can be obtained from this.

Theorem 3.4. *Let Γ be a linearly spanned graph with n vertices. Then Γ generates B_n with $\frac{(n-1)(n-2)}{2}+k$ positive relations described as in the proof below where Γ has $n+k-1$ edges.*

Proof. Choose a spanning tree T of Γ. Then a linearly spanned tree T generates B_n with $\frac{(n-1)(n-2)}{2}$ positive relations as in Lemma 3.3. When each edge α in $\Gamma - T$ are added, a circuit is formed and the circuit give a new positive relation as follows:

1. If α forms a circuit with no intersection with other edges as in Figure 10, add the positive relation
$$\alpha\beta_1\beta_2\cdots\beta_{l-1} = \beta_1\beta_2\cdots\beta_\ell$$
where we regard that the circuit bounds a polygonal disk by cutting open all edges inside the circuit as in Figure 10.

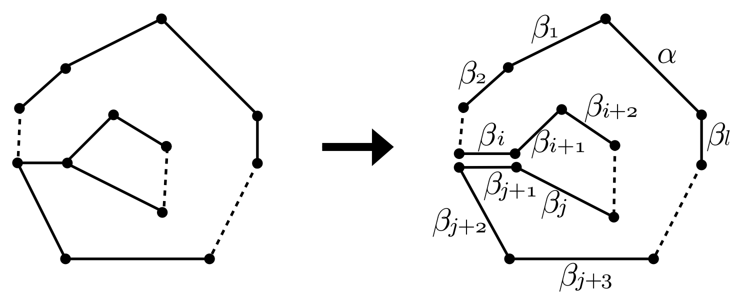

FIGURE 10.

2. If α forms a circuit with some intersections with other edges as in Figure 11, add the positive relation
$$\begin{aligned}&\beta_{11}\beta_{12}\cdots\beta_{1l_1}\beta_{2l_2}\alpha\beta_{2(l_2-1)}\cdots\beta_{22}\\ = \;&\beta_{12}\cdots\beta_{1l_1}\beta_{2l_2}\alpha\beta_{2(l_2-1)}\cdots\beta_{22}\beta_{21}\end{aligned}$$

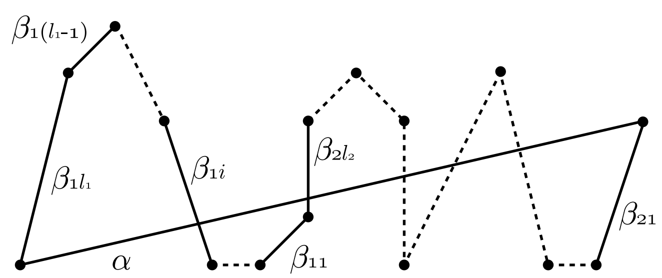

FIGURE 11.

□

4. EMBEDDING PROBLEM

Two positive words U, V in a positive presentation will be said to be *positively equivalent* if they are identically equal or they can be transformed into each other through a sequence of positive words such that each word of the sequence is obtained from the preceding one by a single direct application of the defining relations. And we will write

$U \doteq V$ if U and V are positively equivalent. Given a positive finite presentation $\langle X \mid R \rangle$ of B_n, let B_n^+ be the free semigroup generated by X modulo R. If any two equivalent positive words U and V are positively equivalent, then we say that the semigroup B_n^+ *embeds* in B_n or the presentation $\langle X \mid R \rangle$ has the *embedding property*. A set X of generators is said to have the *embedding property* if a presentation $\langle X \mid R \rangle$ has the embedding property for some finite set R of positive relations. The *embedding problem* of a graph that generates B_n with positive relations is to decide whether the set of generators given by the graph has the embedding property. Thus a graph does not have the embedding property if and only if no finite set of positive relations over the set of generators given by the graph form a presentation with the embedding property.

Garside[7] showed that the Artin presentation has the embedding property and Birman-Ko-Lee[3, 9] showed that the band-generator presentation has the embedding property. We will show that linearly spanned graphs with more than 3 vertices do not have the embedding property except these two presentations. There are 4 possible linearly spanned graphs with 3 vertices as in Figure 12.

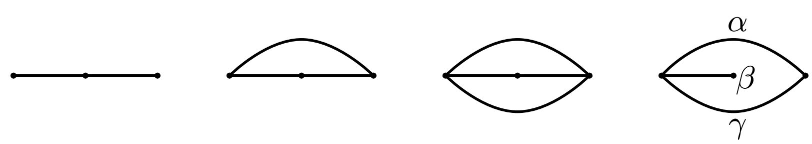

FIGURE 12.

The first two graphs give the Artin and the band-generator presentations of B_3. The last two graphs have multiple edges. We do not know whether these graphs have the embedding property. In particular the following presentation with 4 defining relations:

$$\langle \alpha, \beta, \gamma \mid \alpha\beta\alpha = \beta\alpha\beta, \gamma\beta\gamma = \beta\gamma\beta, \beta^2\gamma = \alpha\beta^2, \beta\gamma\alpha = \gamma\alpha\beta \rangle$$

may have the embedding property. But all known techniques as in [2, 3, 4, 9, 10] fail to apply to this example. We will try to avoid these unknown exceptions in the following discussion by allowing no multiple edges.

A subgraph Γ' of a graph Γ is said to be *full* if every edge in Γ joining two vertices in Γ' is also in Γ'.

Theorem 4.1. *Let Γ' be a connected full subgraph of a linearly spanned graph Γ. If a graph Γ has the embedding property and there is a circle C such that C contains Γ' inside and all vertices in $\Gamma - \Gamma'$ lie outside, then Γ' also has the embedding property.*

Proof. Let X and X' be the set of generators given by Γ and Γ', respectively and let R be a finite set of positive relations on X such that $\langle X \mid R \rangle$ has the embedding property. Let R' be the set of relations on

X' which is a "full" subset of R in the sense that R' contains all relations in R written on X'. We will prove by contradiction that $\langle X' \mid R' \rangle$ has the embedding property. Suppose that there exists a pair of positive words U, V on X' such that U, V are equivalent in the braid group but are not positively equivalent under R'. Since $\langle X|R \rangle$ has the embedding property, $U \doteq V$ under R. Thus there is a sequence of positive words $W_1, \dots, W_k$ over X such that $U \doteq W_1 \doteq \cdots \doteq W_k \doteq V$ under R and each positive equivalence is obtained by one direct application of relations in R. The sequence must contain at least one positive word, say W_i, that is not written solely on X', otherwise $U \doteq V$ under R'. In the view of Lemma 3.3 and Theorem 3.4, it is impossible that R contains any relation $W = W'$ such that W' is a positive word over X' and W is a positive word over the edges that are not incident to any vertex of Γ'. Furthermore Γ' is a full subgraph of Γ. Thus W_i must contain an edge that joins a vertex v in Γ' to a vertex w not in Γ'. Let β be such an occurrence that comes last in W_i. Since Γ' is connected, the vertex v is joined to another vertex u in Γ' by an edge α. As an automorphism of the punctured plane, U does not change the circle C because the edges for U never touch C. On the other hand we will show W_i must change C and this is a contradiction because U and W_i are isotopic as automorphisms of the punctured disk. The automorphism W_i is the composition of the counterclockwise half twists along edges in W_i. Then the counterclockwise half twist along β creates an intersection x of C with the edge α. The intersection x can disappear only via a clockwise half twist along an edge incident at either u or v as in Figure 13. But all of half twists in W_i are counterclockwise because W_i is a positive word over edges,

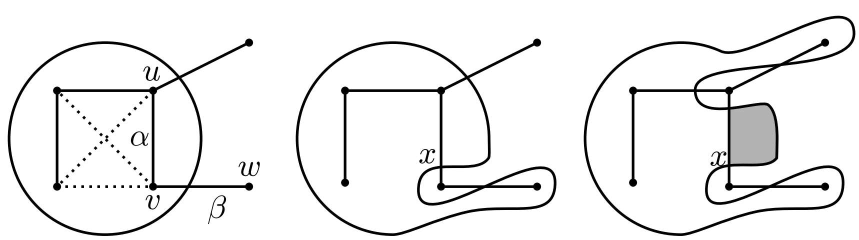

FIGURE 13.

□

We think the above theorem also holds when we replace the condition "linearly spanned" by "connected".

Lemma 4.2. *A linearly spanned graph Γ with 4 vertices has at most one intersection among its edges.*

Proof. An intersection between two adjacent edges always creates a pseudo face that must contain a vertex and so there is no intersection

between adjacent edges in a linearly spanned graph. Suppose Γ has more than one intersection. Figure 14 shows a typical situation with one intersection and another intersection x. The edge α must join vertices v_1 and v_3, otherwise x is an intersection between two adjacent edges. One can easily check that all possibilities of completing α create a pseudo face containing at least a vertex.

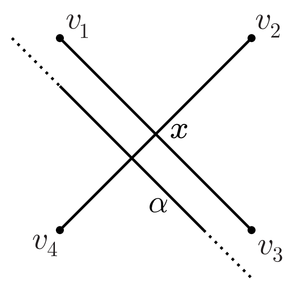

FIGURE 14.

□

Lemma 4.3. *Let Γ be a linearly spanned graph and X be the set of generators given by edges of Γ. Suppose that for each $k = 1, 2, \dots,$ there are positive words $W \equiv \alpha V \beta$ and $W' \equiv \alpha' V' \beta'$ over X for $\alpha, \alpha', \beta, \beta'$ in X such that*

(i) *$W = W'$;*
(ii) *the word length $|W| = |W'| = k + c$ for some constant c;*
(iii) *$\alpha V \neq \alpha' P$ and $V\beta \neq Q\beta'$ for any positive words P, Q over X*

Then the graph Γ does not have the embedding property.

Proof. Let $\langle X \mid R \rangle$ be any positive finite presentation of the braid group. The hypothesis (iii) implies that any shorter positive relation than $W = W'$ itself can not make W positively equivalent to W'. Choose a large k such that $W = W'$ is longer than any relation in R. Then W is not positively equivalent to W' over R. □

The following theorem completely determines when a linearly spanned graph with 4 vertices and no multiple edges has the embedding property. In the proof given below, $P(\alpha_1, \dots, \alpha_k)$ or $Q(\alpha_1, \dots, \alpha_k)$ will denote a positive word over the generators $\alpha_1, \dots, \alpha_k$.

Theorem 4.4. *Among all linearly spanned graphs with 4 vertices and no multiple edges, only two of them, the Artin graph and the inner complete graph as in Figure 15, have the embedding property.*

Proof. The graphs in Figure 15 give the Artin presentation and the band-generator presentation. So they have the embedding property.

In order to show that all other linearly spanned graphs with 4 vertices do not have the embedding property, we appeal to Lemma 4.3. To

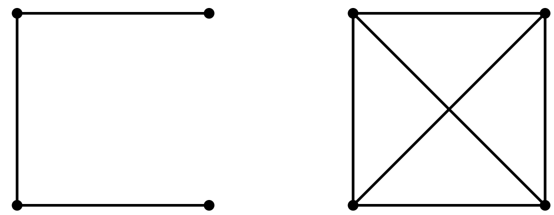

FIGURE 15.

check the hypothesis (iii) of the lemma, we use the fact that the band-generator presentation has the embedding property so that the positive equivalence in the presentation is the same as the equivalence in the braid group. We also utilize the left and right cancellation theorem and the left and right canonical forms in the band-generator presentation in [3, 9].

In the view of Lemma 4.2, the graphs being considered are divided into two types: planar graphs and graphs with one intersection among edges.

Planar graphs. Planar graphs with 4 vertices can be divided further into three types: graphs containing neither a triangle nor a rectangle, graphs containing at least a rectangle, and graphs containing at least a triangle but no rectangle, where a triangle (or a rectangle) is non-degenerate, that is, must have 3 (or 4, respectively) vertices and must contain no other vertices inside.

I. Graphs containing neither a triangle nor a rectangle. This type is further divided into two types (i) and (ii).

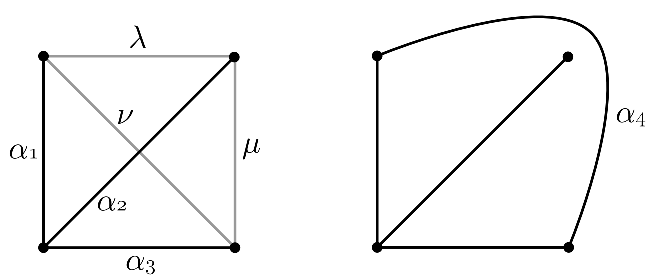

FIGURE 16.

(i) Graphs containing a vertex adjacent to all of remaining three vertices. We may have two possible graphs as in Figure 16 up to equivalence. For each of these two graphs, let $W = \alpha_1\alpha_2\alpha_3^k\alpha_1$ and $W' = \alpha_2\alpha_3^k\alpha_1\alpha_2$. We will show that W and W' satisfy Lemma 4.3. Add the edges λ, μ, ν so that these edges together with $\alpha_1, \alpha_2, \alpha_3$ form an inner-complete graph. Then

$$\alpha_1\alpha_2\alpha_3^k\alpha_1 = \alpha_2\lambda\alpha_3^k\alpha_1 = \alpha_2\alpha_3^k\lambda\alpha_1 = \alpha_2\alpha_3^k\alpha_1\alpha_2$$

and so $W = W'$.

Suppose $\alpha_1\alpha_2\alpha_3^k = \alpha_2 P$ for some positive word P in the band-generator presentation. Then $U \doteq \lambda\alpha_3^k$ by the left cancellation in the band-generator presentation. But $\lambda\alpha_3^k$ may only start with λ or α_3 which commute. Thus P can not be written over $\alpha_1, \alpha_2, \alpha_3$. Similarly $\alpha_2\alpha_3^k\alpha_1 \neq Q(\alpha_1, \alpha_2, \alpha_3)\alpha_2$. Thus W satisfies (iii) of Lemma 4.3.

Due to the triangular relation $\lambda\mu = \mu\alpha_4 = \alpha_4\lambda$, W can not be equivalent to a positive word over $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ that contains α_4 since W is not positively equivalent to a positive word in the band-generator presentation that contains the subword $\lambda\mu$. Thus W, W' also satisfy the hypothesis of Lemma 4.3 over $\alpha_1, \alpha_2, \alpha_3, \alpha_4$.

(ii) Graphs containing no vertex adjacent to all of remaining three vertices. All possible graphs except the Artin graph have multiple edges as in Figure 17.

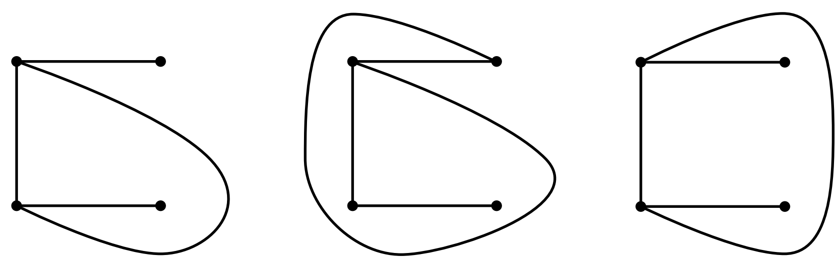

FIGURE 17.

II. Graphs containing at least a rectangle. We may have two possible graphs as in Figure 18 up to equivalence. For each of these five graphs, let $W = \alpha_1\alpha_2^k\alpha_3$ and $W' = \alpha_3\alpha_4^k\alpha_1$.

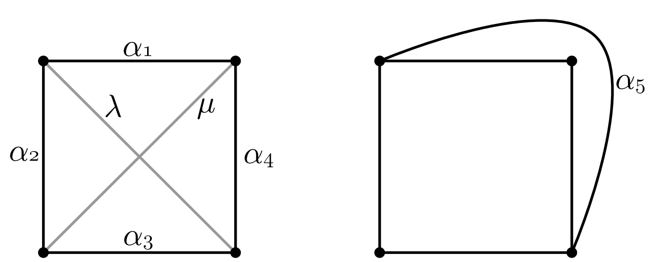

FIGURE 18.

Add the edges λ, μ so that these edges together with $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ form an inner-complete graph. Then

$$\alpha_1\alpha_2^k\alpha_3 = \alpha_1\alpha_2^{k-1}\alpha_3\lambda = \alpha_1\alpha_3\lambda^k = \alpha_3\alpha_1\lambda^k = \alpha_3\alpha_4\alpha_1\lambda^{k-1} = \alpha_3\alpha_4^k\alpha_1$$

and so $W = W'$.

Suppose $\alpha_1\alpha_2^k = \alpha_2 P$ for some positive word P in the band-generator presentation. Then $P \doteq \mu\alpha_2^{k-1}$ by the left cancellation in the band-generator presentation. Thus P can not be written over $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. Similarly $\alpha_2^k\alpha_3 \neq Q(\alpha_1, \dots, \alpha_4)\alpha_1$. Thus W satisfies (iii) of Lemma 4.3.

Due to the triangular relation $\alpha_1\alpha_4 = \alpha_4\alpha_5 = \alpha_5\alpha_1$, W can not be equivalent to a positive word over $\alpha_1, \dots, \alpha_5$ that contains α_5 since W is not positively equivalent to a positive word in the band-generator presentation that contains the subword $\alpha_1\alpha_4$. Thus W, W' also satisfy the hypothesis of Lemma 4.3 over $\alpha_1, \dots, \alpha_5$.

III. Graphs containing at least a triangle but no rectangle. In this case, graphs are divided two types by the number of triangles.
(i) Graphs containing one triangle. We have one possible graph as in Figure 19 up to equivalence. Let $W = \alpha_1\alpha_2\alpha_3\alpha_4\alpha_1\alpha_2^k\alpha_3\alpha_4$ and $W' = \alpha_2\alpha_3^k\alpha_4\alpha_1\alpha_2\alpha_3\alpha_4\alpha_1$.

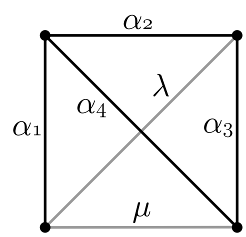

FIGURE 19.

Add the edges λ, μ so that these edges together with $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ form an inner-complete graph. Then

$$\begin{aligned}
\alpha_1\alpha_2\alpha_3\alpha_4\alpha_1\alpha_2^k\alpha_3\alpha_4 &= \alpha_1\alpha_2\alpha_3\alpha_1\mu\alpha_2^k\alpha_3\alpha_4 = \alpha_1\alpha_2\alpha_1\alpha_3\mu\alpha_2^k\alpha_3\alpha_4\\
&= \alpha_2\alpha_1\alpha_2\alpha_3\mu\alpha_2^k\alpha_3\alpha_4 = \alpha_2\alpha_1\alpha_2\alpha_3\alpha_2^k\mu\alpha_3\alpha_4\\
&= \alpha_2\alpha_1\alpha_3^k\alpha_2\alpha_3\mu\alpha_3\alpha_4 = \alpha_2\alpha_1\alpha_3^k\alpha_2\mu\alpha_3\mu\alpha_4\\
&= \alpha_2\alpha_1\alpha_3^k\mu\alpha_2\alpha_3\mu\alpha_4 \ = \alpha_2\alpha_1\alpha_3^k\mu\alpha_2\alpha_3\alpha_4\alpha_1\\
&= \alpha_2\alpha_3^k\alpha_1\mu\alpha_2\alpha_3\alpha_4\alpha_1 = \alpha_2\alpha_3^k\alpha_4\alpha_1\alpha_2\alpha_3\alpha_4\alpha_1
\end{aligned}$$

and so $W = W'$.

Let $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_1\alpha_2^k\alpha_3 = \alpha_2 P$. Then $P = \alpha_1\alpha_2\alpha_3\mu\alpha_2^k\alpha_3 \doteq \alpha_1\alpha_2\alpha_3\alpha_2^k\alpha_3\lambda$. The right canonical form of $\alpha_1\alpha_2\alpha_3\alpha_2^k\alpha_3$ is $\alpha_3^{k-3}(\alpha_1\alpha_3)\alpha_4(\alpha_3\alpha_2)(\alpha_3\alpha_2)$. So $\alpha_1\alpha_2\alpha_3\alpha_2^k\alpha_3$ may finish with $\alpha_2, \alpha_3, \alpha_4$ over the band-generators and so λ or μ must appear in P. Similarly $\alpha_2\alpha_3\alpha_4\alpha_1\alpha_2^k\alpha_3\alpha_4 \neq Q(\alpha_1, \dots, \alpha_4)\alpha_1$.
(ii) Graphs containing two triangles. In this case triangles must share at least 1 edge since we consider only 4 vertices. However if two triangles share 2 edges then it must contain a multiple edge. Thus we have two possible graphs as in Figure 20. For these two graphs, let $W = \alpha_2\alpha_3^k\alpha_5\alpha_2\alpha_3$ and $W' = \alpha_4\alpha_1^k\alpha_5\alpha_4\alpha_1$.

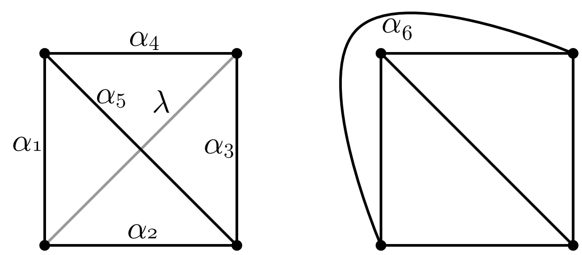

FIGURE 20.

Add the edges λ so that these edges together with $\alpha_1, \dots, \alpha_5$ form an inner-complete graph. Then

$$\begin{aligned} \alpha_2\alpha_3^k\alpha_5\alpha_2\alpha_3 &= \lambda^k\alpha_2\alpha_5\alpha_2\alpha_3 = \lambda^k\alpha_2\alpha_5\alpha_3\lambda = \lambda^k\alpha_2\alpha_4\alpha_5\lambda \\ &= \lambda^k\alpha_4\alpha_2\alpha_5\lambda = \alpha_4\alpha_1^k\alpha_2\alpha_5\lambda = \alpha_4\alpha_1^k\alpha_5\alpha_1\lambda \\ &= \alpha_4\alpha_1^k\alpha_5\alpha_4\alpha_1 \end{aligned}$$

and so $W = W'$.

The left canonical form of $\alpha_2\alpha_3^k\alpha_5\alpha_2$ is $(\alpha_2\alpha_3)\alpha_3^{k-1}\alpha_5\alpha_2$ and so it can not start with α_4 over the band-generators. Thus $\alpha_2\alpha_3^k\alpha_5\alpha_2 \neq \alpha_4P(\alpha_1, \dots, \alpha_5)$. The right canonical form of $\alpha_3^k\alpha_5\alpha_2\alpha_3$ is $\alpha_3^k\alpha_5(\alpha_2\alpha_3)$ and so it can not finish with α_1. Thus $\alpha_3^k\alpha_5\alpha_2\alpha_3 \neq Q(\alpha_1, \dots, \alpha_5)\alpha_1$.

Due to the triangular relation $\alpha_1\alpha_4 = \alpha_4\alpha_6 = \alpha_6\alpha_1$, W can not be equivalent to a positive word over $\alpha_1, \dots, \alpha_5$ that contains α_6 since W is not positively equivalent to a positive word in the band-generator presentation that contains the subword $\alpha_1\alpha_4$. Thus W, W' also satisfy the hypothesis of Lemma 4.3 over $\alpha_1, \dots, \alpha_5$.

Graphs with one intersection. The edges α_1 and α_3 that intersect each other are transformed to the diagonals of a rectangle. Then 4 more edges are needed to make an inner-complete graph. We have four distinct types of graphs, depending on how many edges are missing from the inner-complete graph.

I. Three edges are missing. Figure 21 is the only possible graph in this case. Let $W = \alpha_3\alpha_1\alpha_2^k\alpha_3\alpha_1\alpha_2$ and $W' = \alpha_2\alpha_3\alpha_1\alpha_2^k\alpha_3\alpha_1$.

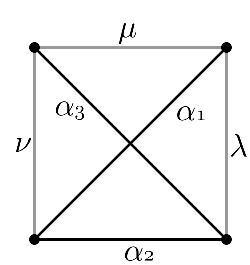

FIGURE 21.

Add the edges λ, μ, ν so that these edges together with $\alpha_1, \alpha_2, \alpha_3$ form an inner-complete graph. Then

$$\begin{aligned} \alpha_3\alpha_1\alpha_2^k\alpha_3\alpha_1\alpha_2 &= \alpha_3\alpha_1\alpha_2^k\alpha_3\lambda\alpha_1 = \alpha_3\alpha_1\alpha_2^k\mu\alpha_3\alpha_1 \\ &= \alpha_3\alpha_1\mu\alpha_2^k\alpha_3\alpha_1 = \alpha_3\nu\alpha_1\alpha_2^k\alpha_3\alpha_1 = \alpha_2\alpha_3\alpha_1\alpha_2^k\alpha_3\alpha_1 \end{aligned}$$

and so $W = W'$.

Let $\alpha_2\alpha_3\alpha_1\alpha_2^k\alpha_3 = \alpha_3 P$. Then $P \doteq \nu\alpha_1\alpha_2^k\alpha_3$. To remove ν, $\alpha_1\alpha_2^k\alpha_3$ must start with α_2. But the left canonical form of $\alpha_1\alpha_2^k\alpha_3$ is $(\alpha_1\alpha_2)(\alpha_2\alpha_3)\nu^{k-2}$ and so it may start only with $\alpha_1, \alpha_2, \lambda$. Thus $\alpha_2\alpha_3\alpha_1\alpha_2^k\alpha_3 \neq \alpha_3 P(\alpha_1\alpha_2\alpha_3)$. Similarly $\alpha_1\alpha_2^k\alpha_3\alpha_1\alpha_2 \neq Q(\alpha_1\alpha_2\alpha_3)\alpha_1$.

II. Two edges are missing. In this case, there are two possible graphs as in Figure 22 and Figure 23. For the graph in Figure 22, let $W = \alpha_4\alpha_3\alpha_2^{k+1}\alpha_3$ and $W' = \alpha_1\alpha_2^2\alpha_1^k\alpha_4$.

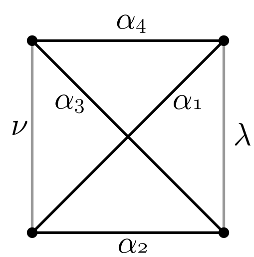

FIGURE 22.

Add the edges λ, ν so that these edges together with $\alpha_1, \ldots, \alpha_4$ form an inner-complete graph. Then

$$\begin{aligned} \alpha_4\alpha_3\alpha_2^{k+1}\alpha_3 &= \lambda\alpha_4\alpha_2^{k+1}\alpha_3 = \lambda\alpha_2^k\alpha_4\alpha_2\alpha_3 = \lambda\alpha_2^k\alpha_4\nu\alpha_2 \\ &= \lambda\alpha_2^k\alpha_1\alpha_4\alpha_2 = \lambda\alpha_2^k\alpha_1\alpha_2\alpha_4 = \lambda\alpha_1\alpha_2\alpha_1^k\alpha_4 = \alpha_1\alpha_2^2\alpha_1^k\alpha_4 \end{aligned}$$

and so $W = W'$.

The left canonical form of $\alpha_4\alpha_3\alpha_2^{k+1}$ is $(\alpha_4\alpha_3)\alpha_2^{k+1}$ and so it can not start with ν over the band-generators. Thus $\alpha_4\alpha_3\alpha_2^{k+1} \neq \alpha_1 P(\alpha_1, \ldots, \alpha_4)$. The right canonical form of $\alpha_3\alpha_2^{k+1}\alpha_3$ is $\alpha_3\alpha_2^k(\alpha_2\alpha_3)$ and so it can not finish with α_4. Thus $\alpha_3\alpha_2^{k+1}\alpha_3 \neq Q(\alpha_1, \ldots, \alpha_4)\alpha_4$.

For the graph in Figure 23, let $W = \alpha_2\alpha_3\alpha_4^k\alpha_2$ and $W' = \alpha_3\alpha_4^k\alpha_2\alpha_3$. Then W, W' satisfy Lemma 4.3 by using a similar argument as for Figure 16.

III. One edge is missing. The only possible graph is Figure 24. Let $W = \alpha_4\alpha_5^k\alpha_2\alpha_3$ and $W' = \alpha_1\alpha_4\alpha_5^k\alpha_2$

$$\alpha_4\alpha_5^k\alpha_2\alpha_3 = \alpha_4\alpha_5^k\lambda\alpha_2 = \alpha_4\lambda\alpha_5^k\alpha_2 = \alpha_1\alpha_4\alpha_5^k\alpha_2$$

and so $W = W'$

The left canonical form of $\alpha_4\alpha_5^k\alpha_2$ is itself and so it can not start with ν over the band-generators. Thus $\alpha_4\alpha_5^k\alpha_2 \neq \alpha_1 P(\alpha_1, \ldots, \alpha_5)$.

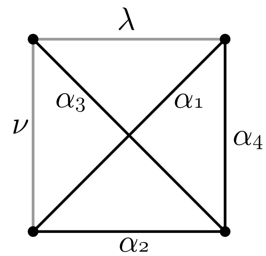

FIGURE 23.

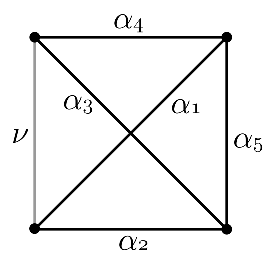

FIGURE 24.

Let $\alpha_5^k\alpha_2\alpha_3 \doteq Q\alpha_2$ then $Q \doteq \alpha_5^k\nu$ and α_5, ν commute, so ν can not be removed. Thus $\alpha_5^k\alpha_2\alpha_3 \neq Q(\alpha_1, \dots, \alpha_5)\alpha_2$.

IV. No edges are missing. If we add any edge to the inner-complete graph, multiple edges are created. Thus only the inner-complete graph is eligible.

□

Theorem 4.5. *Among all linearly spanned graphs without multiple edges, only the Artin graphs and the inner-complete graphs have the embedding property.*

Proof. We have already discussed about linearly spanned graphs with 3 vertices. Let Γ be a linearly spanned graph with more than 3 vertices that has the embedding property. First we choose a connect full subgraph Γ' with 4 vertices from Γ so that there is a separating circle satisfying the hypothesis of Theorem 4.1. Choose 4 vertices that form a connected subtree in a spanning tree of Γ. Take the full subgraph with these 4 vertices. If there is no other vertices in faces of this full subgraph, then this is a desired full subgraph. If there is other vertices in the faces of this full subgraph and none of them is adjacent to the chosen 4 vertices, then there is an edge-path starting at a vertex v on a face and ending at one of the 4 vertices such that the edge-path intersect the full subgraph of the 4 vertices and so v is contained in a pseudo face. Since Γ is linearly spanned, this can not happen. Thus at least one of vertices on faces, say w, is adjacent to one of the 4 vertices. Then we have the less number of unwanted vertices contained in faces of a new connected full subgraph that is obtained by replacing

one of the 4 vertices by w. By repeating this process, we eventually obtain a connect full subgraph Γ' with 4 vertices v_1, v_2, v_3, v_4 that can be separated by a circle from other vertices of Γ. Theorem 4.1 and Theorem 4.4 say that Γ' must be either the Artin graph or the inner-complete graph with 4 vertices. Choose a vertex v_5 in $\Gamma - \Gamma'$ that is adjacent to one of v_1, v_2, v_3, v_4. If Γ' is an inner-complete graph, then each full subgraph with the 4 vertices consisted of v_5 and any three vertices from v_1, v_2, v_3, v_4 must be an inner-complete graph by Theorem 4.1 and Theorem 4.4. Thus the full subgraph with the 5 vertices v_1, v_2, v_3, v_4, v_5 is an inner-complete graph. By repeating this process, Γ itself eventually becomes an inner-complete graph.

We now suppose that Γ' is the Artin graph with 4 vertices v_1, v_2, v_3, v_4 where v_1, v_4 are of valency 1 and the other vertices are of valency 2. Choose a vertex v_5 in $\Gamma - \Gamma'$ that is adjacent to any one of v_1, v_2, v_3, v_4. By Theorem 4.1 and Theorem 4.4, v_5 can be adjacent to either one or both of vertices v_1, v_4. If v_5 is adjacent to both vertices, we stop the process. If v_5 is adjacent to one vertex, say v_4, then $v_1, \dots, v_5$ form an Artin graph and we repeat this process. Eventually we see that Γ must contain one of the following two types of graphs as a full subgraph separated by a circle as in Theorem 4.1 unless Γ itself is an Artin graph. The proof will be completed when we show that both types of graphs do not have the embedding property.

***m*-Gon.** The graph of this type is depicted in Figure 25. Let $W = \alpha_1\alpha_2\cdots\alpha_i^k\cdots\alpha_{m-1}$ and $W' = \alpha_2\cdots\alpha_i^k\cdots\alpha_m$. Then one can show that W, W' satisfy the hypothesis of Lemma 4.3 by a similar but longer argument as for Figure 18. Thus Γ does not have the embedding property

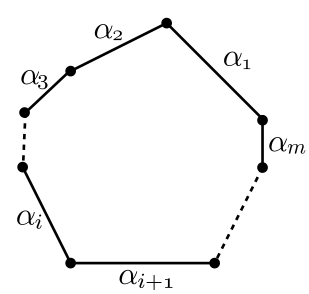

FIGURE 25.

Pseudo *m*-gon. The graph of this type is depicted in Figure 26. Let

$$W = \alpha_2\alpha_3\cdots\alpha_m\alpha_1\alpha_2^k\alpha_3\cdots\alpha_{m-1}\alpha_m\alpha_1\alpha_2\cdots\alpha_{m-2}$$

and

$$W' = \alpha_3\alpha_4\cdots\alpha_m\alpha_1\alpha_2^k\alpha_3\cdots\alpha_{m-1}\alpha_m\alpha_1\alpha_2\cdots\alpha_{m-1}$$

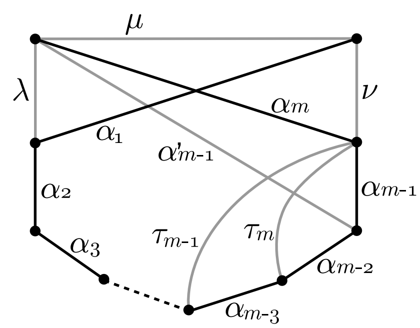

FIGURE 26.

Add supplementary edges $\lambda, \mu, \nu.\alpha'_{m-1}, \tau_{m-1}, \tau_m$ as in Figure 26 to form a subgraph of the inner-complete graph with $m+1$ vertices. Then

$$\begin{aligned}
&\alpha_2\alpha_3\cdots\alpha_m\alpha_1\alpha_2^k\alpha_3\cdots\alpha_{m-1}\alpha_m\alpha_1\alpha_2\cdots\alpha_{m-2}\\
&\quad = \alpha_3\alpha_4\cdots\alpha_m\lambda\alpha_1\alpha_2^k\alpha_3\cdots\alpha_{m-1}\alpha_m\alpha_1\alpha_2\cdots\alpha_{m-2}\\
&\quad = \alpha_3\alpha_4\cdots\alpha_m\alpha_1\mu\alpha_2^k\alpha_3\cdots\alpha_{m-1}\alpha_m\alpha_1\alpha_2\cdots\alpha_{m-2}\\
&\quad = \alpha_3\alpha_4\cdots\alpha_m\alpha_1\alpha_2^k\alpha_3\cdots\alpha_{m-1}\mu\alpha_m\alpha_1\alpha_2\cdots\alpha_{m-2}\\
&\quad = \alpha_3\alpha_4\cdots\alpha_m\alpha_1\alpha_2^k\alpha_3\cdots\alpha_{m-1}\alpha_m\nu\alpha_1\alpha_2\cdots\alpha_{m-2}\\
&\quad = \alpha_3\alpha_4\cdots\alpha_m\alpha_1\alpha_2^k\alpha_3\cdots\alpha_{m-1}\alpha_m\alpha_1\alpha_2\cdots\alpha_{m-1}
\end{aligned}$$

and so $W = W'$.

We now show that

$$\alpha_2\alpha_3\cdots\alpha_m\alpha_1\alpha_2^k\alpha_3\cdots\alpha_{m-1}\alpha_m\alpha_1\alpha_2\cdots\alpha_{m-3} \neq \alpha_3 P(\alpha_1, \dots, \alpha_m).$$

Suppose that

$$\alpha_2\alpha_3\cdots\alpha_m\alpha_1\alpha_2^k\alpha_3\cdots\alpha_{m-1}\alpha_m\alpha_1\alpha_2\cdots\alpha_{m-3} = \alpha_3 P.$$

Then

$$\begin{aligned}
P &\doteq \alpha_4\cdots\alpha_m\lambda\alpha_1\alpha_2^k\alpha_3\cdots\alpha_{m-1}\alpha_m\alpha_1\alpha_2\cdots\alpha_{m-3}\\
&\doteq U_m\tau_m
\end{aligned}$$

where $U_m \equiv \alpha_4\cdots\alpha_m\alpha_1\alpha_2^k\alpha_3\cdots\alpha_{m-1}\alpha_m\alpha_1\alpha_2\cdots\alpha_{m-3}$. Due to the triangular relation $\alpha_{m-1}\tau_m = \alpha_{m-2}\alpha_{m-1}$, the positive word U_m must end with α_{m-1} in order for τ_m to disappear. We will show that U_m can not end with α_{m-1} by an induction on $m \geq 3$. Clearly $U_3 = \alpha_1\alpha_2^k\alpha_3$ can not end with α_2 in the band-generator presentation.

$$\begin{aligned}
U_m &\doteq \alpha_4\cdots\alpha_{m-1}\alpha_m\alpha_1\alpha_2^k\alpha_3\cdots\alpha_{m-1}\alpha_m\alpha_1\alpha_2\cdots\alpha_{m-3}\\
&\doteq \alpha_4\cdots\alpha_{m-2}\alpha'_{m-1}\alpha_{m-1}\alpha_1\alpha_2^k\alpha_3\cdots\alpha_{m-1}\alpha_m\alpha_1\alpha_2\cdots\alpha_{m-3}\\
&\doteq \alpha_4\cdots\alpha_{m-2}\alpha'_{m-1}\alpha_1\alpha_{m-1}\alpha_2^k\alpha_3\cdots\alpha_{m-1}\alpha_m\alpha_1\alpha_2\cdots\alpha_{m-3}\\
&\doteq \alpha_4\cdots\alpha_{m-2}\alpha'_{m-1}\alpha_1\alpha_2^k\alpha_3\cdots\alpha_{m-1}\alpha_{m-2}\alpha_{m-1}\alpha_m\alpha_1\alpha_2\cdots\alpha_{m-3}\\
&\doteq \alpha_4\cdots\alpha_{m-2}\alpha'_{m-1}\alpha_1\alpha_2^k\alpha_3\cdots\alpha_{m-2}\alpha_{m-1}\alpha_{m-2}\alpha_m\alpha_1\alpha_2\cdots\alpha_{m-3}
\end{aligned}$$

$$\begin{aligned}
&\doteq \alpha_4 \cdots \alpha_{m-2}\alpha'_{m-1}\alpha_1\alpha_2^k\alpha_3 \cdots \alpha_{m-2}\alpha_{m-1}\alpha_m\alpha_1\alpha_2 \cdots \alpha_{m-4}\alpha_{m-2}\alpha_{m-3} \\
&\doteq \alpha_4 \cdots \alpha_{m-2}\alpha'_{m-1}\alpha_1\alpha_2^k\alpha_3 \cdots \alpha_{m-2}\alpha'_{m-1}\alpha_{m-1}\alpha_1\alpha_2 \cdots \alpha_{m-4}\alpha_{m-2}\alpha_{m-3} \\
&\doteq \alpha_4 \cdots \alpha_{m-2}\alpha'_{m-1}\alpha_1\alpha_2^k\alpha_3 \cdots \alpha_{m-2}\alpha'_{m-1}\alpha_1\alpha_2 \cdots \alpha_{m-4}\alpha_{m-1}\alpha_{m-2}\alpha_{m-3} \\
&\doteq U_{m-1}\alpha_{m-1}\alpha_{m-2}\alpha_{m-3}
\end{aligned}$$

It is easy to check that $U_{m-1}\alpha_{m-1}\alpha_{m-2}\alpha_{m-3}$ ends with α_{m-1} if and only if $U_{m-1}\alpha_{m-1}\alpha_{m-2}$ ends with α_{m-1} if and only if U_{m-1} ends with α_{m-2} in the band-generator presentation. By the induction hypothesis, U_{m-1} can not end with α_{m-2}. Similarly we have that

$$\alpha_3 \cdots \alpha_m\alpha_1\alpha_2^k\alpha_3 \cdots \alpha_{m-1}\alpha_m\alpha_1\alpha_2 \cdots \alpha_{m-2} \neq Q(\alpha_1, \ldots, \alpha_m)\alpha_{m-1}.$$

Consequently W, W satisfy the hypothesis of Lemma 4.3. □

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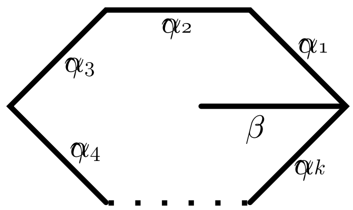
α_1
α_2
α_3
α_4
α_k
β